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Entanglement and magnetic order

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Abstract

In recent years quantum statistical mechanics have benefited of cultural interchanges with quantum information science. There is much evidence that quantifying the entanglement allows a fine analysis of many relevant properties of many-body quantum systems. Here we review the relation between entanglement and the various types of magnetic order occurring in interacting spin systems.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In one of the most influential papers for the foundation of quantum physics (Schrödinger 1935), entanglement was recognized as not ‘one but rather *the* characteristic trait of quantum mechanics’. Although such kind of non-local correlations was thoroughly explored for the analysis of the conceptual fundamentals of quantum mechanics (Bell 1987, Peres 1993), the interest in understanding the properties of entangled states has received an impressive boost with the advent of quantum information. In fact it was understood that non-local correlations are responsible for the enhanced efficiency of quantum protocols (Nielsen and Chuang 2000). In some other cases, quantum teleportation just to mention an important example, quantum correlations are a necessary ingredient. Entanglement can thus be considered a resource for quantum information processing. The need for defining the amount of resources required to implement a given protocol has lead to a fertile line of research in quantum information aiming at the quantification of entanglement (Horodecki *et al* 2007). To this end, necessary criteria for any entanglement measure to be fulfilled have been elaborated and lead to the notion of an entanglement monotone (Vidal 2000). There is a fairly clear scenario in the case of bipartite systems. The multipartite counterpart is much more intricate.

The large body of knowledge developed in the characterization of entanglement has found important applications in areas that, less than a decade ago, were quite distant from quantum information. This is the case of quantum statistical mechanics. Traditionally, the characterization of many-body systems has been carried on through the study of different physical quantities (as the magnetization in magnetic systems) and their correlations. Very little attention was paid to the structure of their quantum state and in particular to their amount of entanglement. The research at the border between these areas is rapidly evolving and has led to numerous interesting results. The promise of this new interdisciplinary field is twofold: firstly, to improve our understanding of strongly correlated systems, beyond the current state of the art; and secondly, to control and manipulate quantum correlations to our convenience, ultimately with the hope of making an impact on the computation schemes to solve ‘hard problems’ in computer science.

Methods developed in quantum information have proven to be extremely useful in the analysis of the state of many-body systems. At the same time experience built up over the years in condensed matter is helping in finding new protocols for quantum computation and communication. The cross-fertilization of quantum information with statistical mechanics should not, however, come as a surprise. Quantum computers are themselves many-body systems, and the main difference from traditional solid-state systems is that a quantum computer can be controlled and operated under non-equilibrium conditions. It is therefore natural to profit from the best instruments developed so far in the two disciplines for the understanding of quantum complex systems, being quantum computers or condensed matter systems. In this paper we deal with a particular aspect of this research area: we will focus on interacting spin systems on a lattice and describe the relations between entanglement and magnetic ordering. A more detailed discussion on these issues can be found in Amico *et al* (2008) and Vedral (2008).

A complete characterization of entanglement in a many-body system is hopeless. The number of possible ways in which the system can be partitioned explodes on increasing the number of elementary constituents (namely spins). Nevertheless, judicious choices have made possible to highlight interesting properties of many-body correlations. Numerous contributions in this volume consider a bipartition in which the system is divided into two distinct regions. If the total system is in a pure state, as when the system is in its ground state, then a measure of the entanglement between the two regions is given by the von Neumann entropy associated with the reduced density matrix (of one of the two regions). This is not however the only possible choice; our paper will briefly review other ways to quantify entanglement whose properties contribute to our understanding of many-body systems.

In the case of bipartite entanglement one can consider the quantum correlation between two given spins after having traced out the rest of the system. In this case the entanglement between the two selected sites can be quantified by the concurrence (Wootters 2001). The study of two-site entanglement, as we will briefly describe in this paper, allows us to detect quite well the presence of quantum phase transition in the phase diagram. A different approach to entanglement in many-body systems arises from the quest to swap or transmute different types of multipartite entanglement into pairwise entanglement between two parties by means of generalized measures on the rest of the system. In a system of interacting spins on a lattice this maximizes the entanglement between two spins by performing measurements on all the others. With this aim, the concept of localizable entanglement has been introduced in Verstraete *et al* (2004a) and Popp *et al* (2005).

The structure of a many-body system’s quantum state is by far much richer than that captured by bipartite entanglement. In the multipartite case the grounds for quantitative predictions are less firm because of the exceptional difficulty of the problem. Nevertheless,

there are a number of very interesting results already available. Among all we mention here the bounds which have been derived on the ground-state energy (Gühne *et al* 2005) which allow us to discriminate among different n -particle quantum correlations. The idea of deriving bounds for macroscopic quantities (as for example the ground-state energy) is quite a powerful method to characterize entanglement in many-body system and is related to the concept of entanglement witness. As an interesting connection between statistical mechanics and quantum information, it turns out that in many cases thermodynamic quantities such as the magnetization or the susceptibility (in the case of an interacting spin model) behave as entanglement witness (Tóth 2005, Wiesniak *et al* 2005), thus providing a way to detect entanglement experimentally. Finally, we would like to mention that the natural dynamics of condensed matter systems may be important to detect bound entangled states (Patanè *et al* 2007, Ferraro *et al* 2008) which are, in several cases, difficult to be realized artificially.

Topic of this paper is to complement the different papers in this volume by describing various measures of entanglement, other than the block entropy, which were used to characterize the equilibrium and dynamical properties of entanglement in spin systems (see also the article by Latorre and Riera in the same volume). Interacting spin models (Auerbach 1998, Schollwöck *et al* 2004) provide a paradigm to describe a wide range of many-body systems. They account for the effective interactions in a variety of very different physical contexts ranging from high energy to nuclear physics. In condensed matter besides describing the properties of magnetic compounds, they capture several aspects of high-temperature superconductors, quantum Hall systems, heavy fermions, just to mention few important examples. Of course interacting spins are central to quantum information processing (Nielsen and Chuang 2000).

The paper is organized as follows. In the next section we list the spin models that will be considered thereafter. Then we give a brief overview of the entanglement measures that are currently used to analyze many-body systems. In section 4 we discuss the main outcomes of the uses of this entanglement measure applied to the spin models. The conclusions and a possible outlook will be presented in section 5.

2. Model systems

A large class of relevant models of interacting spin on a d -dimensional lattice can be described by the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{i,j} [J_x^{(ij)} S_i^x S_j^x + J_y^{(ij)} S_i^y S_j^y + J_z^{(ij)} S_i^z S_j^z] - h^z \sum_i S_i^z. \quad (1)$$

In equation (1) S_i^α ($\alpha = x, y, z$) are spin-1/2 operators defined on the i th site of a d -dimensional lattice. The ground state of equation (1) is, in general, a highly entangled state. Nonetheless, there exists a point where the ground state is indeed a classical state, factorized in the direct space (Kurmman *et al* 1982, Roscilde *et al* 2005b, Giampaolo *et al* 2009). Such a phenomenon occurs at a precise value of the field h_f that was obtained in any-dimensional bipartite lattice and for finite-range exchange interaction (Giampaolo *et al* 2009) and even in the presence of frustration (Giampaolo *et al* 2009); for factorization in ferrimagnets see Rezai *et al* 2009. Despite the similarities with the saturation phenomenon occurring in ferromagnets in external magnetic fields, it was proved that the factorization of the ground state is due to a fine tuning of the control parameters, *within the magnetically ordered phase*. Interestingly, the results obtained so far indicate that the factorization point is a precursor of the quantum phase transition. We also mention that analysis of the ground state for finite-size systems

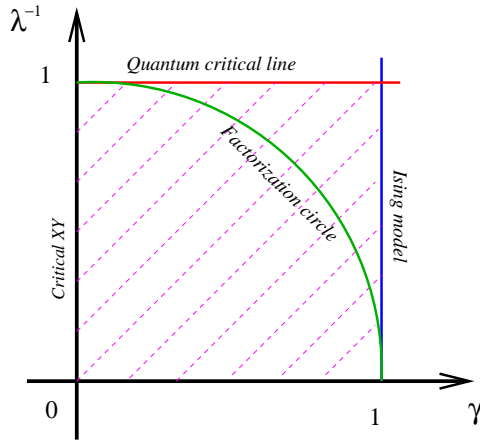


Figure 1. The zero-temperature phase diagram of the one-dimensional anisotropic XY model in a transverse field. Along the quantum critical line the model identifies the Ising universality class with indices $z = \nu = 1$; in the hatched area (with $\gamma > 0$) the system displays a long-range order in the x - y spin components. The critical XY regime coincides with that of the XXZ model for $\Delta = 0$. The factorization of the ground state occurs along the circle $\lambda^{-1} = \sqrt{1 - \gamma^2}$.

demonstrated that the factorization can be viewed as transition between ground states of different parity (Rossignoli *et al* 2008, Giorgi 2009).

For nearest neighbors, the previous Hamiltonian defines the XYZ anisotropic Heisenberg model. In this case the exchange couplings are commonly parameterized as $J_x = J(1 + \gamma)$, $J_y = J(1 - \gamma)$ and $J_z = 2J\Delta$. A positive (negative) exchange coupling J favors antiferromagnetic (ferromagnetic). In one dimension the model defined by Hamiltonian (1) is exactly solvable in several important cases. This is particularly interesting in the analysis of entanglement because quantum effects are particularly pronounced in low dimensions.

Whenever $\Delta = 0$, the (quantum anisotropic XY) Hamiltonian can be diagonalized by first applying the Jordan–Wigner transformation and then performing a Bogoliubov transformation (Lieb *et al* 1961, Pfeuty 1970, Barouch and McCoy 1971). The quantum Ising model corresponds to $\gamma = 1$ while the (isotropic) XX -model is recovered for $\gamma = 0$. In the isotropic case the model possesses an additional symmetry resulting in the conservation of the total magnetization along the z -axis. The properties of the Hamiltonian are governed by the dimensionless coupling constant $\lambda = J/2h$. The phase diagram is sketched in figure 1. In the interval $0 < \gamma \leq 1$ the system undergoes a second-order quantum phase transition at the critical value $\lambda_c = 1$. The order parameter is the magnetization in the x -direction, $\langle S^x \rangle$, different from zero for $\lambda > 1$. In the phase with broken symmetry the ground state has a twofold degeneracy reflecting a global phase flip symmetry of the system. The magnetization along the z -direction, $\langle S^z \rangle$, is different from zero for any value of λ , but displays singular behavior in its first derivative at the transition point. In the whole interval $0 < \gamma \leq 1$ the transition belongs to the Ising universality class. For $\gamma = 0$ the quantum phase transition is of the Berezinskii–Kosterlitz–Thouless type. In several cases the evaluation of entanglement requires the determination of the average magnetization $M_I^\alpha(t) = \langle \psi | S_I^\alpha(t) | \psi \rangle$ and of the equal-time correlation functions $g_{lm}^{\alpha\beta}(t) = \langle \psi | S_I^\alpha(t) S_m^\beta(t) | \psi \rangle$. These correlators have been calculated for this class of models in the case of thermal equilibrium (Lieb *et al* 1961, Pfeuty 1970, Barouch and McCoy 1971). These can be recasted in the form of Toeplitz determinants in the equilibrium case and can be expressed as a sum of Pfaffians in certain non-equilibrium situations (Amico and Osterloh 2004).

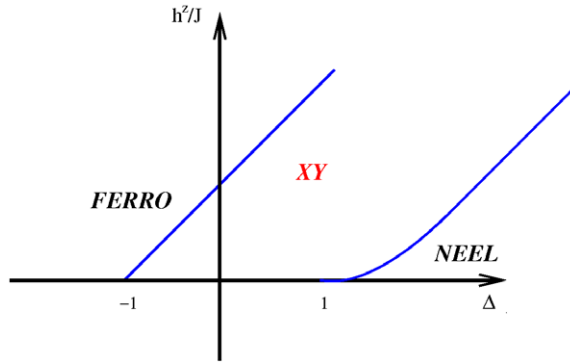


Figure 2. Zero-temperature phase diagram of the spin-1/2 XXZ model in one dimension. The XY phase is characterized by power-law decay of the xy correlations. The Neel and the XY phases are separated by a line of second-order phase transitions; at $\Delta = 1$ the transition is of Berezinskii–Kosterlitz–Thouless. The ferromagnetic and XY phases are separated by first-order phase transitions due to simple level crossings; the onset to the ferromagnetic phase occurs through the saturation phenomenon.

In the case in which $\gamma = 0$ and for any value of Δ the model is referred to as the XXZ model. The two isotropic points $\Delta = 1$ and $\Delta = -1$ describe the antiferromagnetic and ferromagnetic chains, respectively (see the phase diagram in figure 2). The one-dimensional XXZ Heisenberg model can be solved exactly by the Bethe Ansatz technique (see e.g. Takahashi 1999) and the correlation functions can be expressed in terms of certain determinants (see Bogoliubov *et al* (1993) for a review). Correlation functions, especially for intermediate distances, are in general difficult to evaluate, although important steps in this direction have been made (Kitanine *et al* 1999, Göhmann and Korepin 2000, Boos *et al* 2008). The zero-temperature phase diagram of the XXZ model in a zero magnetic field shows a gapless phase in the interval $-1 \leq \Delta < 1$ with power law decaying correlation functions (Mikeska and Pesch 1977, Tonegawa 1981). Outside this interval the excitations are gapped. The two phases are separated by a Berezinskii–Kosterlitz–Thouless phase transition at $\Delta = 1$ while at $\Delta = -1$ the transition is of the first order. In the presence of the external magnetic field a finite energy gap appears in the spectrum. The universality class of the transition is not affected, as a result of the conservation of the total spin- z component (Takahashi 1999).

Another interesting case of the Hamiltonian in equation (1) is when each spin interacts with all the other spins in the system with the same coupling strength

$$\mathcal{H} = -\frac{J}{2} \sum_{ij} [S_i^x S_j^x + \gamma S_i^y S_j^y] - \sum_i \mathbf{h}_i \cdot \mathbf{S}_i.$$

For site-independent magnetic field $h_i^\alpha = h^\alpha \forall i, \alpha = x, y, z$, this model is known as the Lipkin–Meshkov–Glick (LMG) model (Lipkin *et al* 1965, Meshkov *et al* 1965a, 1965b). In this case the dynamics of the system can be described in terms of a collective spin $S_\alpha = \sum_j S_j^\alpha$. At $J^2\gamma = 4h^z$ the Hamiltonian manifests a supersymmetry (Unanyan and Fleischhauer 2003). The phase diagram depends on the parameter $\lambda = J/2h^z$. For a ferromagnetic coupling ($J > 0$) and $h^x = h^y = 0$ the system undergoes a second-order quantum phase transition at $\lambda_c = 1$, characterized by mean-field critical indices (Botet *et al* 1982). For $h^y = 0, h^z < 1$ and $\gamma = 0$ the model exhibits a first-order transition at $h^x = 0$ (Vidal *et al* 2006) while for an antiferromagnetic coupling and $h^y = 0$ a first-order phase transition at $h^z = 0$ occurs for any γ 's. The model Hamiltonian defined above embraces an important class of interacting

fermion systems with pairing force interaction (like the BCS model). Both the LMG- and the BCS-type models can be solved exactly by Bethe ansatz (Richardson 1963, Richardson and Sherman 1964).

We end this very brief overview with spin-1 systems which were originally considered to study the quantum dynamics of magnetic solitons in antiferromagnets with single-ion anisotropy (Mikeska 1995). In one dimension, half-integer and integer spin chains have very different properties (Haldane 1983a, 1983b). We will see that the typical ground state of such models displays characteristic features in its entanglement content. The long-range order that is established in the ground state of systems with half-integer spin (Lieb *et al* 1961) may be washed out for integer spins. In this latter case, the system has a gap in the excitation spectrum. A paradigm model of interacting spin-1 systems is

$$H = \sum_{i=0}^N \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \beta (\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2. \quad (2)$$

The lack of long-range order arises because of the presence of zero as an eigenvalue of S_i^z ; the corresponding eigenstates represent a spin configuration that can move freely in the chain, ultimately disordering the ground state of the system, experiencing a gap with the lowest energy excitation (Mikeska 1995, Gomez-Santos 1991). The so-called string order parameter was proposed to capture the resulting ‘floating’ Néel order, made of alternating spins $|\uparrow\rangle, |\downarrow\rangle$ with strings of $|0\rangle$ ’s in between (den Nijs and Rommelse 1989)

$$O_{\text{string}}^\alpha = \lim_{R \rightarrow \infty} \left\langle S_i^\alpha \left(\prod_{k=i+1}^{i+R-1} e^{i\pi S_k^\alpha} \right) S_{i+R}^\alpha \right\rangle. \quad (3)$$

The ground state of physical systems described by Hamiltonians of the form of equation (2) has been studied in great detail (Schollwöck *et al* 2004). Different phase transitions have been found between antiferromagnetic phases, Haldane phases and a phase characterized by a large density of vanishing weights ($S_i^z = 0$) along the chain. Some features of the phenomenology leading to the destruction of the antiferromagnetic order can be put on a firm ground for $\beta = 1/3$ (AKLT model), where the ground state of the Hamiltonian in equation (2) is known exactly (Affleck *et al* 1988). In this case it was proved that the ground state is constituted by a sea of nearest-neighbor valence bond states, separated from the first excitation by a finite gap with exponentially decaying correlation functions.

3. Entanglement measures

The study of quantum correlations in many-body systems depends heavily on the impressive progress that has been achieved in the theory of entanglement quantification. The new ingredient that makes the many-body case very appealing is the rich variety of ways in which the system can be partitioned into subsystems. Comprehensive overviews of entanglement measures can be found in Bengtsson and Życzkowski (2006), Bruß (2002), Eisert (2006), Horodecki *et al* (2007), Plenio and Vedral (1998), Plenio and Virmani (2007), Vedral (2002) and Wootters (2001); it is however convenient to recall some of the entanglement measures that are routinely used to characterize many-body systems. Important requirements for an entanglement measure are that it should be invariant under local unitary operations; it should be continuous and, furthermore, additive when several identical copies are considered.

Most of the work done on entanglement in many-body systems deals with the bipartite case. A pure bipartite state is not entangled if, and only if, it can be written as a tensor product of pure states of the parts. It can be demonstrated that reduced density matrices can be

decomposed by exploiting the Schimdt decomposition: $\rho_{B/A} = \sum_i \alpha_i^2 |\psi_{B/A,i}\rangle\langle\psi_{B/A,i}|$. Since only product states lead to pure reduced density matrices, a measure for their mixedness points a way toward quantifying entanglement. One can thus use a suitable function of α_i given by the Schmidt decomposition to quantify the entanglement. Remarkably enough the von Neumann entropy $S(\rho_{B/A}) = -\sum_i \alpha_i^2 \log(\alpha_i^2)$ can *quantify* the entanglement encoded in $\rho_{B/A}$. We point out that an infinite class of entanglement measures can be constructed for pure states. In fact by tracing out one of two qubits in the state, the corresponding reduced density matrix ρ_A contains only a single independent parameter: its eigenvalue is $\leq 1/2$. This implies that each monotonic function $[0, 1/2] \mapsto [0, 1]$ of this eigenvalue can be used as an entanglement measure. A relevant example is the (one-) tangle (Coffman *et al* 2000) $\tau_1[\rho_A] = 4\det \rho_A$. By expressing ρ_A in terms of spin expectation values, it follows that $\tau_1[\rho_A] = \frac{1}{4} - (\langle S^x \rangle^2 + \langle S^y \rangle^2 + \langle S^z \rangle^2)$ where $\langle S^\alpha \rangle = \text{tr}_A(\rho_A S^\alpha)$ and $S^\alpha = \frac{1}{2}\sigma^\alpha$, σ^α $\{\alpha = x, y, z\}$ being the Pauli matrices. For a pure state of two qubits it can be shown that τ_1 is equivalent to the concurrence C (Hill and Wootters 1997, Wootters 1998) for pure states of two qubits. The von Neumann entropy can be expressed as a function of the (one-) tangle $S[\rho_A] = h[(1 + \sqrt{1 - \tau_1[\rho_A]})/2]$ where $h(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$ is the binary entropy.

Subsystems of a many-body (pure) state will generally be in a mixed state. In this case different ways of quantifying entanglement can be realized. Three important representatives are the entanglement cost E_C , the distillable entanglement E_D (both defined in Bennett *et al* (1996a)) and the entanglement of formation E_F (Bennett *et al* 1996c). In the following we concentrate on the entanglement of formation. The conceptual difficulty behind its calculation lies in the infinite number of possible decompositions of a density matrix. Therefore, even knowing how to quantify bipartite entanglement in pure states, we cannot simply apply this knowledge to mixed states in terms of an average over the mixtures of pure-state entanglement. It turns out that the correct procedure is to take the minimum over all possible decompositions. This conclusion can be drawn from the requirement that entanglement must not increase on average by means of local operations including classical communication. The entanglement of formation of a state ρ is therefore defined as

$$E_F(\rho) := \min \sum_j p_j S(\rho_{A,j}), \tag{4}$$

where the minimum is taken over all realizations of the state $\rho_{AB} = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, and $S(\rho_{A,j})$ is the von Neumann entropy of the reduced density matrix $\rho_{A,j} := \text{tr}_B |\psi_j\rangle\langle\psi_j|$. For systems of two qubits, an analytic expression for E_F does exist and it is given by

$$E_F(\rho) = - \sum_{\sigma=\pm} \frac{\sqrt{1 + \sigma C^2(\rho)}}{2} \ln \frac{\sqrt{1 + \sigma C^2(\rho)}}{2} \tag{5}$$

where $C(\rho)$ is the the so-called concurrence (Wootters 1998, 2001) defined as

$$C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \tag{6}$$

where $\lambda_1^2 \geq \dots \geq \lambda_4^2$ are the eigenvalues of $R \equiv \sqrt{\rho} \tilde{\rho} \sqrt{\rho} = \sqrt{\rho} (\sigma^y \otimes \sigma^y) \rho^* (\sigma^y \otimes \sigma^y) \sqrt{\rho}$ ($*$ indicates the complex conjugation). As the entanglement of formation is a monotonic function of the concurrence, also C itself or its square τ_2 —called also the two-tangle—can be used as entanglement measures. The concurrence C and the tangle τ_1 both range from 0 (no entanglement) to 1. By virtue of (6), the concurrence in a spin-1/2 chain can be computed in terms of the two-point spin correlation functions. As an example (that is relevant for the present paper) we consider a case where the model has a parity symmetry, it is translational invariant and the Hamiltonian is real. In this case the concurrence reads

$$C_{ij} = 2 \max \{0, C_{ij}^I, C_{ij}^{II}\}, \tag{7}$$

where $C_{ij}^I = |g_{ij}^{xx} + g_{ij}^{yy}| - \sqrt{(1/4 + g_{ij}^{zz})^2 - M_z^2}$ and $C_{ij}^{II} = |g_{ij}^{xx} - g_{ij}^{yy}| + g_{ij}^{zz} - 1/4$, with $g_{ij}^{\alpha\alpha} = \langle S_i^\alpha S_j^\alpha \rangle$ and $M_z = \langle S^z \rangle$. A state with dominant fidelity of parallel and antiparallel Bell states is characterized by dominants C^I and C^{II} , respectively. This was shown in Fubini *et al* (2006), where the concurrence was expressed in terms of the fully entangled fraction as defined in Bennett *et al* (1996c).

The importance of the tangle and the concurrence is due to the monogamy of entanglement which was expressed in terms of an inequality in Coffman *et al* (2000) for the case of three qubits. This inequality has been proved to hold also for n -qubits system (Osborne and Verstraete 2006). In the case of many qubits it reads $\sum_{j \neq i} C_{ij}^2 \leq \tau_{1,i}$. The so-called residual tangle, $\tau_{1,i} - \sum_{j \neq i} C_{ij}^2$, is a measure for multipartite entanglement not stored in pairs of qubits only.

Another measure of entanglement we mention is the relative entropy of entanglement (Vedral *et al* 1997). It can be applied to any number of qubits in principle (or any dimension of the local Hilbert space). It is formally defined as $E(\sigma) := \min_{\rho \in \mathcal{D}} S(\sigma \| \rho)$, where $S(\sigma \| \rho) = \text{tr} \sigma [\ln \sigma - \ln \rho]$ is the quantum relative entropy. This relative entropy of entanglement quantifies the entanglement in σ by its distance from the set \mathcal{D} of separable states. The main difficulty in computing this measure is to find the disentangled state closest to ρ . This is in general a non-trivial task, even for two qubits. In the presence of certain symmetries—which is the case for e.g. eigenstates of certain models—an analytical access is possible. In these cases, the relative entropy of entanglement becomes a very useful tool. The relative entropy reduces to the entanglement entropy in the case of pure bipartite states; this also means that its, so-called, *convex roof extension* (Uhlmann 1998) coincides with the entanglement of formation, and is readily deduced from the concurrence (Wootters 1998).

It is important to realize that not just the quantification of many-party entanglement is a difficult task; it is an open problem to tell in general, whether a state of n parties is separable or not. It is therefore of great value to have a tool that is able to merely certify if a certain state is entangled. An entanglement witness W is an operator that is able to detect entanglement in a state. The basic idea is that the expectation value of the witness W for the state ρ under consideration exceeds certain bounds only when ρ is entangled. An expectation value of W within this bound however does not guarantee that the state is separable. Nonetheless, this is a very appealing method also from an experimental point of view, since it is sometimes possible to relate the presence of the entanglement to the measurement of few observables. Simple geometric ideas help to explain the witness operator W at work. Let \mathcal{T} be the set of all density matrices and let \mathcal{E} and \mathcal{D} be the subsets of entangled and separable states, respectively. The convexity of \mathcal{D} is a key property for witnessing entanglement. The entanglement witness is then an operator defining a hyper-plane which separates a given entangled state from the set of separable states. The main scope of this geometric approach is then to optimize the witness operator (Lewenstein *et al* 2000) or to replace the hyper-plane by a curved manifold, tangent to the set of separable states (Gühne 2004). We have the freedom to choose W such that $\text{tr}(\rho_D W) \leq 0$ for all disentangled states $\rho_D \in \mathcal{D}$. Then $\text{tr}(\rho W) > 0$ implies that ρ is entangled. Entanglement witnesses are a special case of a more general concept, namely that of positive maps. These are injective superoperators on the subset of positive operators. When we now think of superoperators acting non-trivially only on part of the system (on operators that act non-trivially only on a sub-Hilbert space), then we may ask the question whether a positive map on the subspace is also positive when acting on the whole space. Maps that remain positive also on the extended space are called completely positive maps. Positive but not completely positive maps are important for entanglement theory. Indeed it can be shown (Horodecki *et al* 1996) that state ρ_{AB} is entangled if and only if a positive map Λ exists

(not completely positive) such that $(\mathbb{1}_A \otimes \Lambda_B)\rho_{AB} < 0$. For a two-dimensional local Hilbert space the situation simplifies and in a system of two qubits the lack of complete positivity in a positive map is due to a partial transposition. This partial transposition clearly leads to a positive operator if the state is a tensor product of the parts. In fact, also the opposite is true: a state of two qubits ρ_{AB} is separable if and only if $\rho_{AB}^{T_B} \geq 0$, that is, its partial transposition is positive. This is very simple to test and it is known as the Peres–Horodecki criterion (Peres 1996, Horodecki *et al* 1996). The properties of entangled states under partial transposition lead to a measure of entanglement known as the *negativity*. The negativity N_{AB} of a bipartite state is defined as the absolute value of the sum of the negative eigenvalues of $\rho_{AB}^{T_A}$. The *logarithmic negativity* is then defined as

$$E_N = \log_2 2(2N_{AB} + 1). \quad (8)$$

For bipartite states of two qubits, $\rho_{AB}^{T_A}$ has at most one negative eigenvalue (Sanpera *et al* 1998). For general multipartite and higher local dimension this is only a sufficient condition for the presence of entanglement. There exist entangled states with a positive partial transpose (PPT) known as bound entangled states (Acin *et al* 2001, Horodecki *et al* 1998). The existence of bound entangled states ultimately limits the possibility of exploiting the violation of Bell inequalities as a measure of entanglement.

Let us briefly introduce the concept of bound entanglement. Such kind of entanglement can be recognized in terms of the so-called distillation protocol, a non-trivial procedure to optimize the extraction of Bell states from a mixture of entangled states (Bennett *et al* 1996b). The natural question then is: Can any entangled state be actually distilled? The answer is yes for bipartite and qubit states (Horodecki *et al* 1997). For multipartite entangled states and higher dimensional local Hilbert spaces a much more complex scenario emerges. In these cases examples of entangled states have been provided that cannot be distilled to maximally entangled states between the parties of the system, not even with an asymptotically infinite supply of copies of the state. Such a demoted form of entanglement was termed as bound entanglement. PPT entangled states were found first in Horodecki *et al* (1998). The existence of bipartite (with higher dimensional local Hilbert spaces) NPT bound entanglement has not been excluded yet. This question has important implication on the additivity property of the distillable entanglement: if affirmative, entangled states could be generated from a mixture of non-distillable states. Besides its speculative interest, it was demonstrated that bound entanglement can be activated in several quantum information and teleportation tasks to ‘restore’ the singlet fidelity of a given state (see Horodecki *et al* (2007)).

Multipartite bound entangled states exist that are not fully separable but contain entanglement between each of their parties that cannot be distilled. Given that violation of PPT is necessary for distillation, a feature of multipartite bound entanglement is related to the ‘incomplete separability’ of the state (see Dür and Cirac (2000)). A tripartite system A–B–C, for example, is separable with respect to the partition A–BC and B–AC and non-separable with respect to C–AB. The ‘incomplete separability’ is a sufficient condition for a state to have bound entanglement since the three qubits are entangled and no maximally entangled state can be created between any of the parties by LOCC. For example, no entanglement can be distilled between C and A because no entanglement can be created with respect to the partition A–BC by LOCC. In the next section the feature of incomplete separability will be exploited to detect multipartite bound entanglement in spin systems. The nature of correlations in bound entangled states is peculiar having both quantum and classical features. Therefore, if it is true that all entangled states violate Bell inequality, the vice versa has not proven (see also Popescu (1995) and Gisin (1996)). Bipartite bound entangled states seem not violating Bell inequalities (Masanes 2006) (but some examples of multipartite bound entangled state violating Bell-type

inequalities were found (Dür 2001)). For both the bipartite and multipartite entangled states there is a strong believe that necessary and sufficient condition for local realism is that the state satisfies the Peres criterium (Peres 1999, Werner and Wolf 2000, 2001).

As already mentioned, a classification of multipartite entanglement is still missing. Nevertheless, there are several quantities serving as indicators for multipartite entanglement when the whole system is in a pure state. The entropy of entanglement is an example for such a quantity and several works use multipartite measures constructed from and related to it (see e.g. Coffman *et al* (2000), Meyer and Wallach (2002), Barnum *et al* (2003), Scott (2004), de Oliveira *et al* (2006a), Love *et al* (2007)). These measures give indication on a global correlation without discerning among the different entanglement classes encoded in the state of the system. The geometric measure of entanglement quantifies the entanglement of a pure state through the minimal distance of the state from the set of pure product states (Vedral *et al* 1997, Wei and Goldbart 2003)

$$E_g(\Psi) = -\log_2 \max_{\Phi} |\langle \Psi | \Phi \rangle|^2 \quad (9)$$

where the maximum is on all product states Φ . It is zero for separable states and rises up to unity for e.g. the maximally entangled n -particle GHZ states. The difficult task in its evaluation is the maximization over all possible separable states and of course the convex roof extension to mixed states. Despite these complications, a clever use of the symmetries of the problem renders this task accessible by substantially reducing the number of parameters over which the maximization has to be performed. Another example for the collective measures of multipartite is the measures introduced by Meyer and Wallach (2002) and by Barnum *et al* (2003, 2004). In the case of the qubit system the measure of Meyer and Wallach is the average purity (which is the average one-tangle in Coffman *et al* (2000)) of the state (Meyer and Wallach 2002, Brennen 2003, Barnum *et al* 2004)

$$E_{gl} = 2 - \frac{2}{N} \sum_{j=1}^N \text{Tr} \rho_j^2. \quad (10)$$

The notion of generalized entanglement introduced in Barnum *et al* (2003, 2004) relaxes the typically chosen partition into local subsystems in real space. For the state $|\psi\rangle$ it is defined as

$$P_A = \text{Tr}\{[\mathcal{P}_A|\psi\rangle\langle\psi|]^2\} \quad (11)$$

where \mathcal{P}_A is the projection map $\rho \rightarrow \mathcal{P}_A(\rho)$. If the set of observables is defined by the operator basis $\{A_1, A_2, \dots, A_L\}$, then $P_A = \sum_{i=1}^L \langle A_i \rangle^2$ from which the reduction to equation (10) in the case of all local observables is evident. This conceptually corresponds to a redefinition of locality as induced by the distinguished observable set. Finally, we mention the approach pursued in Gühne *et al* (2005) where different bounds on the average energy of a given system were obtained for different types of n -particle quantum correlated states. A violation of these bounds then implies the presence of multipartite entanglement in the system. The starting point of Gühne *et al* is the notion of n -separability and k -producibility which is believed to discriminate particular types of n -particle correlations present in the system. A pure state $|\psi\rangle$ of a quantum systems of N parties is said to be n -separable if it is possible to find a partition of the system for which $|\psi\rangle = |\phi_1\rangle|\phi_2\rangle \cdots |\phi_n\rangle$. A pure state $|\psi\rangle$ can be produced by k -party entanglement (i.e. it is k -producible) if we can write $|\psi\rangle = |\phi_1\rangle|\phi_2\rangle \cdots |\phi_m\rangle$ where $|\phi_i\rangle$ are states of maximally k parties; by definition $m \geq N/k$. It implies that it is sufficient to generate specific k -party entanglement to construct the desired state. Both these indicators for multipartite entanglement are collective, since they are based on the property of a given many-particle state to be factorized into smaller parts. k -separability and k -producibility both do not distinguish between different k -particle entanglement classes (as e.g. the k -particle W-states

and different k -particle graph states (Hein *et al* 2004), like the GHZ state). Another approach is based on the observed relation between entanglement measures and $SL(2, \mathbb{C})$ invariant antilinear operators (Uhlmann 2000, Osterloh and Siewert 2005, 2006, Osterloh and Djokovic 2009). This allows certain sensitivity to different classes of multipartite entanglement (see also Verstraete *et al* (2002), Lamata *et al* (2007), Bastin *et al* (2009), Osterloh (2008)).

We close this section by reviewing how to swap or transmute different types of multipartite entanglement in a many-body system into pairwise entanglement between two parties by means of generalized measures on the rest of the system. In a system of interacting spins on a lattice one could then try to maximize the entanglement between two spins (at positions i and j) by performing measurements on all the others. The system is then partitioned in three regions: the sites i, j and the rest of the lattice. This concentrated pairwise entanglement can then be used e.g. for quantum information processing. A standard example is that of a GHZ state $(1/\sqrt{2})(|000\rangle + |111\rangle)$. After a projective measure in the x -direction on one of the sites such a state is transformed into a Bell state. The concept of *localizable entanglement* has been introduced in Verstraete *et al* (2004a), Popp *et al* (2005). It is defined as the maximal amount of entanglement that can be localized, on average, by doing local measurements in the rest of the system. In the case of N parties, the possible outcomes of the measurements on the remaining $N - 2$ particles are pure states $|\psi_s\rangle$ with corresponding probabilities p_s . The localizable entanglement E_{loc} on the sites i and j is defined as the maximum of the average entanglement over all possible outcome states $|\psi_s\rangle_{ij}$:

$$E_{\text{loc}}(i, j) = \sup_{\mathcal{E}} \sum_s p_s E(|\psi_s\rangle_{ij}) \quad (12)$$

where \mathcal{E} is the set of all possible outcomes $(p_s, |\psi_s\rangle)$ of the measurements, and E represents the chosen measure of entanglement of a pure state of two qubits (e.g. the concurrence). Although very difficult to compute, lower and upper bounds have been found which allow us to deduce a number of non-trivial properties of the state. An upper bound to the localizable entanglement is given by the entanglement of assistance (Laustsen *et al* 2003) obtained from localizable entanglement when also global and joint measurements were allowed on the $N - 2$ spins. A lower bound of the localizable entanglement (Verstraete *et al* 2004a) is fixed by the maximal correlation function between the two parties.

4. Entanglement and magnetic order

The entanglement present in the equilibrium (thermal or ground) state of a quantum system is very sensitive to the underlying collective behavior. This suggests, in the case of spin systems, to analyze the relation between entanglement and magnetic order. We will discuss various aspects of this connection starting from the pairwise entanglement, and we then proceed with the properties of multipartite entanglement. Most of the investigations available in the literature are for one-dimensional systems where exact results are available, and later on we will overview the status in the d -dimensional case.

The body of knowledge acquired so far makes it evident that entanglement in the ground state contains relevant information on zero-temperature phase diagram of the system. We will highlight this relation in two paradigmatic cases when the spin system is either close to quantum phase transitions or to a factorizing field. We will mostly be concerned with an XYZ spin models in an external field.

Range of pairwise entanglement. As we discussed in section 2, in an interacting spin system there exists a particular choice of the coupling constants and the external field for which the

ground state is factorized (Kurmann *et al* 1982, Giampaolo *et al* 2009), i.e. the entanglement vanishes exactly. Several works were devoted to the characterization of the entanglement close to the *factorizing point*. It was demonstrated for one-dimensional spin models in the class *XXZ* that the point at which the state of the system becomes separable marks an exchange of parallel and antiparallel sector in the ground-state concurrence (Fubini *et al* 2006, Amico *et al* 2006a). This change occurs through a global (long-range) reorganization of the state of the system. The range R of the concurrence (defined through the maximum distance between two sites over which it is non-zero) diverges. For the *XY* model it was found that this range is

$$R \propto \left(\ln \frac{1-\gamma}{1+\gamma} \right)^{-1} \ln |\lambda^{-1} - \lambda_f^{-1}|^{-1}. \quad (13)$$

The existence of such a divergence has been confirmed in other one-dimensional systems both for short- (Amico *et al* 2006a, Roscilde *et al* 2004, 2005a) and long-range interactions (Dusuel and Vidal 2005). This divergence suggests, as a consequence of the monogamy of the entanglement (Coffman *et al* 2000, Osborne and Verstraete 2006), that the role of pairwise entanglement is enhanced on approaching the factorizing field (Roscilde *et al* 2004, 2005a, 2005b). Indeed, for the Ising model (i.e. $\gamma = 1$), one finds that in this region the ratio between the two-tangle and the one-tangle tends to 1 (figure 3) (Amico *et al* 2006a).

The diverging entanglement length is particularly intriguing in systems characterized by the topological order. In Son *et al* (2009) the entanglement in the quasi-long-range ordered ground state of the one-dimensional isotropic *XY* model was analyzed. Because of the presence of the characteristic edge states in systems with non-trivial topology, it was found that the quasi-long-range order is traced by entangled states localized at the edges of the system. The range of the concurrence diverges also close to the saturation field for a spin system with inverse-square interaction of the Haldane–Shastry type (Haldane 1988, Shastry 1988), interpolating in a sense between nearest-neighbor and fully connected graph interactions. In particular, in Giuliano *et al* (2009) it was shown that in the absence of external magnetic field the Haldane–Shastry spin system only displays nearest-neighbor entanglement, while by increasing the magnetic field, the bipartite entanglement between spins at greater distance increases, up to a situation where the spin system saturates, entering a fully polarized phase described by a completely separable ground state. Given the special role of the inverse-square interaction in one-dimensional fractional statistics (Comtet *et al* 1999), this study is also relevant for the understanding of the interplay between the statistics and the entanglement.

The range of concurrence does not diverge only at the factorizing field. There are one-dimensional spin systems where the pairwise entanglement has qualitative different features as a function of the distance between the sites. An example is the *long-distance entanglement* observed in Campos Venuti *et al* (2006a). Given a measure of entanglement $E(\rho_{ij})$, Campos Venuti *et al* showed that it is possible that $E(\rho_{ij}) \neq 0$ when $|i - j| \rightarrow \infty$ in the ground state. Long-distance entanglement can be realized in various one-dimensional models as in the dimerized frustrated Heisenberg models or in the AKLT model. For these two models the entanglement is highly non-uniform and it is mainly concentrated in the end-to-end pair of the chain.

Pairwise entanglement and quantum phase transitions. A great number of papers have been devoted to the study of entanglement close to quantum phase transition (QPT). Close to the quantum critical point the system is characterized by a diverging correlation length ξ which is responsible for the singular behavior of different physical observables. The behavior of correlation functions, however, is not necessarily related to the behavior of entanglement. It is worth to stress that the study of entanglement close to quantum critical points does not provide new understanding to the scaling theory of quantum phase transitions. Rather it may

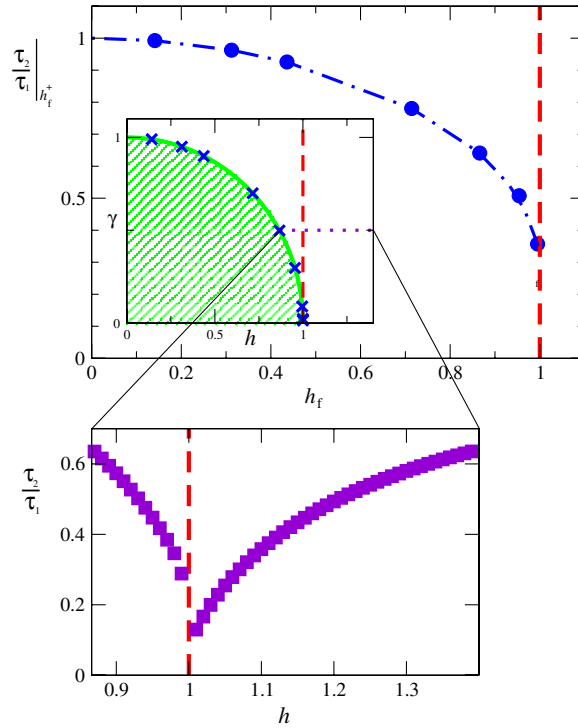


Figure 3. One-tangle entanglement ratio for the XY model: the dashed line indicates the critical field. Upper panel: value of τ_2/τ_1 as $h \rightarrow h_f^+$, as a function of $h_f^{XY} = \sqrt{1 - \gamma^2}$, for different values of γ ; the inset shows the ‘entanglement phase diagram’ in the $h - \gamma$ plane: the shaded (white) area corresponds to antiparallel (parallel) pairwise entanglement and the full line is the separable circle; the crosses indicate the position of the data shown in the main upper panel, while the dotted line corresponds to the x -axis used in the lower panel. Lower panel: τ_2/τ_1 versus h , for $\gamma = 0.5$. (From Amico *et al* (2006b).)

be useful in a deeper characterization of the ground-state wavefunction of the many-body system undergoing a phase transition. In this respect it is important to explore, for instance, how the entanglement depends on the order of the transition, or what is the role of the range of the interaction to establish the entanglement in the ground state. We start by considering exclusively the properties of pairwise entanglement.

Pairwise entanglement close to quantum phase transitions was originally analyzed in Osborne and Nielsen (2002) and Osterloh *et al* (2002) for the Ising model in one dimension. Below we summarize their results in this specific case. The concurrence tends to zero for $\lambda \gg 1$ and $\lambda \ll 1$, and the ground state of the system is fully polarized along the x -axes (z -axes). Moreover, the concurrence is zero unless the two sites are at most next-nearest neighbors; we therefore discuss only the nearest-neighbor concurrence $C(1)$. The concurrence itself is a smooth function of the coupling with a maximum close to the critical point (but not related to any property of the phase transition). The critical properties of the ground state are captured by the derivatives of the concurrence as a function of λ . The results are shown in figure 4. In the thermodynamic limit $\partial_\lambda C(1)$ diverges on approaching the critical value as

$$\partial_\lambda C(1) \sim \frac{8}{3\pi^2} \ln|\lambda - \lambda_c|. \tag{14}$$

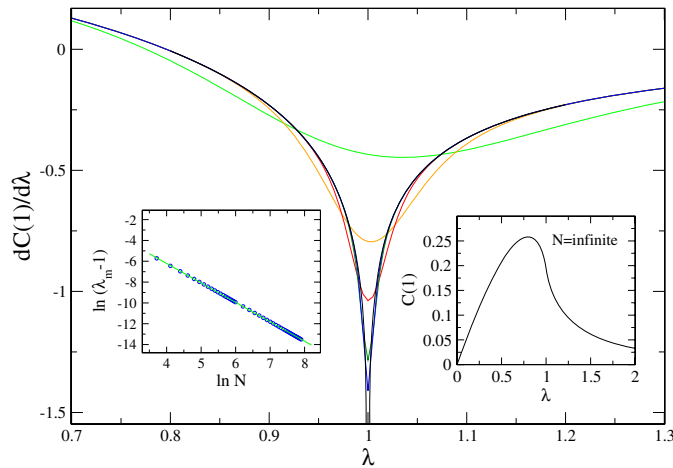


Figure 4. The change in the ground-state wavefunction in the critical region is analyzed considering the derivative of the nearest-neighbor concurrence as a function of the reduced coupling strength. The curves correspond to different lattice sizes. On increasing the system size, the minimum gets more pronounced. Also the position of the minimum changes and tends as (see the left side inset) toward the critical point where for an infinite system a logarithmic divergence is present. The right-hand side inset shows the behavior of the concurrence itself for an infinite system. Reprinted with permission from Osterloh *et al* (2002). Copyright (2002) Nature Publishing Group.

For a finite system the precursors of the critical behavior can be analyzed by means of finite-size scaling of the derivative of the concurrence. Similar results have been obtained for the XY universality class (Osterloh *et al* 2002). Remarkably, although the concurrence describes short-range properties, nevertheless, scaling behavior typical of continuous phase transition emerges.

Over the last years the properties of pairwise entanglement were intensively studied. It was evidenced how it depends on the order of transition and on the universality class of the system. The bulk of results obtained so far can be summarized in a ‘Ehrenfest classification scheme for entanglement’, ultimately arising because of the formal relation between the correlation functions and the entanglement. A way to put this observation on a quantitative ground is provided by a generalized Hohenberg–Kohn theorem (Wu *et al* 2006). Accordingly, the ground-state energy can be considered as a unique function of the expectation values of certain observables. These, in turn, can be related to (the various derivatives of) a given entanglement measure (Wu *et al* 2004, Campos Venuti *et al* 2006b). It was indeed shown that, given an entanglement measure M related to reduced density operators of the system, first-order phase transition are associated with the anomalies of M while second-order phase transitions correspond to a singular behavior of the derivatives of M . Also the quasi-long-range order is captured by the behavior of the pairwise entanglement (Gu *et al* 2003, Son *et al* 2009). Other singularities like those noted in the concurrence for models with three-spin interactions (Yang 2005) are due to the non-analyticity intrinsic in the definition of the concurrence as a minimum of two analytic functions and the constant zero. This was then explicitly shown for the quantum Ising, XXZ and LMG models (Wu *et al* 2006). For the Ising model, for example, the divergence of the first derivative of the concurrence is determined by the non-analytical behavior of $\langle S^x S^x \rangle$ (Wu *et al* 2004). A relevant caveat to this approach is constituted by the uniaxial LMG model in a transverse field (with $h^y = 0$ and $\gamma = 0$) that displays a first-order

QPT for $h^x = 0$. The concurrence is continuous at the transition since it does not depend on the discontinuous elements of the reduced density matrix (Vidal *et al* 2004). The relation between entanglement and criticality was also studied in the spin-1 XXZ with single-ion anisotropy. It was established that the critical anomalies in the entropy experienced at the Haldane-large- D (if an axial anisotropy $D \sum_i (S_i^z)^2$ is added to the Hamiltonian in equation (2)) transition fan out from the singularity of the local-order parameter $\langle (S^z)^2 \rangle$ (Campos Venuti *et al* 2006b). Spontaneous symmetry breaking can influence the entanglement in the ground state. Below the critical field, the concurrence is enhanced by the parity symmetry breaking (Osterloh *et al* 2006). Recently it was demonstrated that such enhancement is particularly pronounced for multipartite entanglement close to the symmetry breaking. This result constitutes a further indication that multipartite, and not bipartite, entanglement plays the main role to establish long-range correlations at the critical points (de Oliveira *et al* 2008).

In higher dimensions nearly all the results were obtained by means of numerical simulations. The concurrence was computed for the two-dimensional quantum XY and XXZ models (Syljuåsen 2003a). The calculations were based on quantum Monte Carlo simulations (Sandvik and Kurkij 1991, Syljuåsen and Sandvik 2002). Although the concurrence for the 2D models results to be qualitatively very similar to the one-dimensional case, it is much smaller in magnitude. The monogamy limits the entanglement shared among the number of neighbor sites. The ground-state entanglement in a two-dimensional XYZ model was analyzed in Roscilde *et al* (2005b) by means of quantum Monte Carlo simulations. The divergence of the derivative of the concurrence at the continuous phase transition, observed in $d = 1$, was confirmed; also in this case the range of the pairwise entanglement extends only to few lattice sites. By studying the one- and the two-tangle of the system, it was proved that the QPT is characterized by a cusp-minimum in the entanglement ratio τ_1/τ_2 . The cusp is ultimately due to the discontinuity of the derivative of τ_1 . The minimum in the ratio τ_1/τ_2 makes it evident that the multipartite entanglement plays an enhanced role in the mechanism driving the phase transition. Moreover, by looking at the entanglement it was found that the ground state can be factorized at certain value of the magnetic field. The existence of the factorizing field in $d = 2$ was proved rigorously for any $2d - XYZ$ model in a bipartite lattice. Unexpectedly enough the relation implying the factorization is very similar to that one found in $d = 1$.

Pairwise entanglement at finite temperature. At finite temperature excitations participate to entanglement that can become non-monotonic on increasing temperature or magnetic field (Arnesen *et al* 2001, Gunlycke *et al* 2001). Concurrence for thermal states was calculated in several situations (Osborne and Nielsen 2002, Wang 2002a, Wang and Zanardi 2002, Tribedi and Bose 2006, Asoudeh and Karimipour 2004, Canosa and Rossignoli 2005, 2006, Rigolin 2004, Zhang and Zhu 2006, Wang and Wang 2006, Zhang and Li 2005). At finite temperatures but close to a quantum critical points, quantum fluctuations are essential to describe the properties of the systems (Sachdev 1999). For illustration let us consider a one-dimensional quantum XY in an external magnetic field. Although such a system cannot exhibit any phase transitions at finite temperature, the very existence of the quantum critical point is reflected in the crossover behavior at $T \neq 0$. According to the standard nomenclature, the renormalized-classical regime evolves into the quantum disordered phase through the so-called *quantum critical region* (Sachdev 1999). In the T - h plane a V-shaped phase diagram emerges, characterized by the crossover temperature customarily defined as $T_{\text{cross}} \doteq |\lambda^{-1} - \lambda_c^{-1}|$. For $T \ll T_{\text{cross}}$ the thermal De Broglie length is much smaller than the average spacing of the excitations; therefore, the correlation functions factorize into two contributions coming from quantum and thermal fluctuations separately. The quantum critical region is characterized by $T \gg T_{\text{cross}}$. Here we are in the first regime and the correlation functions do not factorize. In

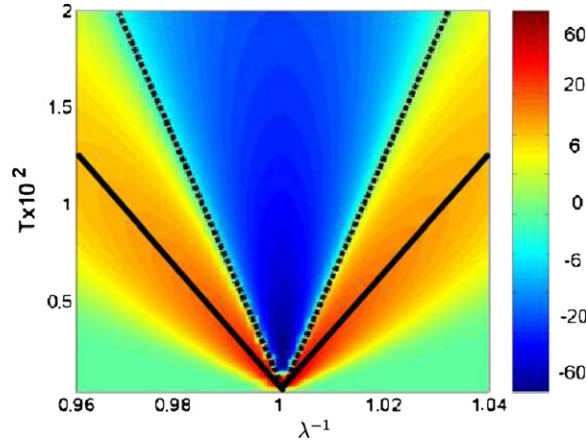


Figure 5. The effect of temperature on the anomalies originated from the critical divergence of the field derivative of $C(R)$ can be measured by $\partial_T[\partial_a C(R)]$. The density plot corresponds to $\gamma = 1$ and $R = 1$. $T = T^*$ and $T = T_M$ are drawn as dashed and thick lines respectively. Maxima below T^* are found at $T_M = \beta T_{\text{cross}}$ with $\beta \sim 0.290 \pm 0.005$ and they are independent of γ and R ; the crossover behavior is enclosed in between the two flexes of $\partial_T[\partial_a C(R)]$ at T_{c1} and T_{c2} ; such values are fixed to $T_{c1} = (0.170 \pm 0.005)T_{\text{cross}}$ and $T_{c2} = (0.442 \pm 0.005)T_{\text{cross}}$ and found to be independent of γ and R . For T smaller than T_{c1} $\partial_T[\partial_a C(R)] \simeq 0$. Scaling properties are inherited in $\partial_T[\partial_a C(R)]$ from $\partial_a C(R)$. (Reprinted with permission from Amico and Patané (2007).) Copyright (2007) by IOP Publishing Ltd.

this regime the interplay between quantum and thermal effects is the dominant phenomenon affecting the physical behavior of the system. Thermal entanglement close to the critical point of the quantum XY models was recently studied by some of us (Amico and Patané 2007). In analogy with the zero-temperature case it was shown that the entanglement sensitivity to thermal and to quantum fluctuations obeys universal $T \neq 0$ scaling laws. The crossover to the quantum disordered and renormalized classical regimes in the entanglement has been analyzed through the study of derivatives of the concurrence $\partial_\lambda C$ and $\partial_T C$. The thermal entanglement results to be very rigid when the quantum critical regime is accessed from the renormalized classical and quantum disordered regions of the phase diagram; such a ‘stiffness’ is reflected in a maximum in $\partial_T C$ at $T \sim T_{\text{cross}}$. The maximum in the derivatives of the concurrence seems a general feature of the entanglement in the crossover regime (see for example Stauber and Guinea (2004, 2006)). Due to the vanishing of the gap at the quantum critical point, in the region $T \gg T_{\text{cross}}$ an arbitrarily small temperature is immediately effective in the system (see figure 5). From the analysis of the quantum mutual information it emerges that the contribution of the classical correlations is negligible in the crossover, thus providing the indication that such a phenomenon is driven solely by the thermal entanglement. It is interesting to study how the existence of the factorizing field h_f affects the thermal pairwise entanglement (vanishing at zero temperature). It results that the two-tangle τ_2 is still vanishing in a region of the h - T plane fanning out from h_f ; therefore, if present, the entanglement in the region must be shared between three or more parties. In contrast to the analysis of the ground state, at finite temperature one cannot characterize the two separate phases of parallel and antiparallel entanglement.

Thermal entanglement witnesses. In some case it is hard to quantify the entanglement in a many-body system. Moreover, it seems in general difficult to relate clear observables to some of the entanglement measures. If one relaxes the requirement of quantifying the entanglement

and asks only to know if a state is entangled or not, then in some important cases there is a very appealing answer in terms of the so-called entanglement witness. Interestingly, it was shown that entanglement witnesses in spin systems can be related to thermodynamic quantities (Tóth 2005, Brukner and Vedral 2004, Wu *et al* 2005, Hide *et al* 2007). For the isotropic XXX or XX Heisenberg model, if the inequality is fulfilled (with U the internal energy, M^z the magnetization)

$$\frac{|U + h^z M^z|}{N|J|} > \frac{1}{4}, \quad (15)$$

then the system is in an entangled state. Once the internal energy and magnetization are calculated, it is possible to verify in which range the parameters of the system and the external temperature entanglement are present. Most important is the fact that these types of inequalities can be verified experimentally. It should be stressed that the analysis based on the entanglement witness could be applied to any model for which we can successfully obtain the partition function. This feature is the main advantage of using the thermodynamic witnesses approach to detecting entanglement. This method for determining entanglement in solids within the models of Heisenberg interaction is useful in the cases where other methods fail due to incomplete knowledge of the system. This is the case when only the eigenvalues but not eigenstates of the Hamiltonian are known (which is the most usual case in solid state physics) and thus no measure of entanglement can be computed. Furthermore, in the cases where we lack the complete description of the systems one can approach the problem experimentally and determine the value of the thermodynamical entanglement witness by performing appropriate measurements. It is important to emphasize that any other thermodynamical function of state could be a suitable witness, such as the magnetic susceptibility or heat capacity (Wiesniak *et al* 2005).

Localizable entanglement. The study of localizable entanglement in spin chains allows us to find a tighter connection between the scales over which entanglement and correlations decay (Verstraete *et al* 2004a, Popp *et al* 2005, 2006). One expects that the procedure of entangling distant sites by a set of local measurements will be less effective as the distance between the two particles increases thus leading to a definition of entanglement length ξ_E . For a translational invariant system ξ_E can be defined in analogy of the standard correlation length:

$$\xi_E^{-1} = - \lim_{|i-j| \rightarrow \infty} \log \frac{E_{\text{loc}}(|i-j|)}{|i-j|}. \quad (16)$$

By definition the entanglement length cannot be smaller than the correlation length, $\xi_E \geq \xi$; therefore, at a second-order phase transition the localizable entanglement length diverges. In addition there may also appear ‘transition points’ associated solely with a divergence in ξ_E . In order to avoid misinterpretations, it must be stressed that the *localizable* ‘classical’ two-point correlations then diverge as well. For the Ising model in a transverse field it can be shown that (Verstraete *et al* 2004b) $\max_{\alpha=x,y,z} |Q_{\alpha}^{ij}| \leq E_{\text{loc}}(i-j) \leq \frac{1}{2} \sum_{\pm} \sqrt{s_{\pm}^{ij}}$ where $s_{\pm}^{ij} = (1 \pm \langle S_i^z S_j^z \rangle)^2 - (\langle S_i^z \rangle \pm \langle S_j^z \rangle)^2$ and $Q_{\alpha}^{ij} = \langle S_i^{\alpha} S_j^{\alpha} \rangle - \langle S_i^{\alpha} \rangle \langle S_j^{\alpha} \rangle$. In this case, the lower bound is determined by the two-point correlation function in the x -direction. In the disordered phase ($\lambda < 1$) the ground state possesses a small degree of entanglement and consequently its entanglement length is finite. The situation changes at the other side of the critical point. Here, although the correlation length is finite, the entanglement length is infinite as asymptotically the correlation tends to a finite values. The divergence of ξ_E indicates that the ground state is a globally entangled state, supporting the general idea that multipartite entanglement is most relevant at the critical point (Osborne and Nielsen 2002,

Roscilde *et al* 2004). The properties of localizable entanglement were further investigated for a spin-1/2 XXZ-chain in Jin and Korepin (2004) and Popp *et al* (2005) as a function of the anisotropy parameter Δ and of an externally applied magnetic field h . The authors used exact results for correlation functions relying on the integrability of the models to find the required bounds. The presence of the anisotropy further increases the lower bound of the localizable entanglement. At the Berezinskii–Kosterlitz–Thouless critical point ($\Delta = 1$) the lower bound of the nearest-neighbor localizable entanglement shows a kink (Popp *et al* 2005). As pointed out by the authors this might have implications in the general understanding of the Berezinskii–Kosterlitz–Thouless phase transitions where the ground-state energy and its derivatives are continuous as well as the concurrence. The localizable entanglement in a two-dimensional XXZ model was discussed as well (Syljuåsen 2003b) by means of quantum Monte Carlo simulations. A lower bound has been determined by studying the maximum correlation function which for $\Delta > -1$ is Q_x , the long-range (power-law) decay of the correlation implying a long-ranged localizable entanglement.

For half-integer spins, gapped non-degenerate ground states are characteristic for systems in a disordered phase (consider paramagnets for example). A finite gap in the excitation spectrum of the system in the thermodynamic limit makes the correlations decaying exponentially. This is the Lieb–Schultz–Mattis theorem establishing that, under general hypothesis, the ground state of a spin system is either unique and gapless or gapped and degenerate (Lieb *et al* 1961) (see Hastings (2004) for recent results). It was a surprise, when Haldane discovered that systems of integer spins can violate this theorem (Haldane 1983a, 1983b). Accordingly the long-range order can be replaced by the so-called hidden order of topological nature. This kind of order is established in the system because certain solitonic type of excitations become gapless for integer spins (Mikeska 1995). This suggests the need to investigate whether the entanglement in the ground state might play some role in establishing the hidden order characteristic for the Haldane phases. An aspect that might be relevant to this aim was recently addressed by studying the localizable entanglement in AKLT models (Verstraete *et al* 2004a). The ground state of this class of models is of the valence bond type. For this case it was demonstrated that a singlet state made of two spins-1/2 located at the ends of the chain can be always realized. This implies that the localizable entanglement is long ranged despite the exponentially decaying correlation (Verstraete *et al* 2004a). Furthermore, the localizable entanglement can be related to the string order parameter. The valence-bond-solid phase order was further studied by looking at the hidden order in chains with more complicated topology. The von Neumann entropy was studied in the spin-1 XXZ model with biquadratic interaction and single-ion anisotropy in Gu *et al* (2006), Wang *et al* (2005) and Campos Venuti *et al* (2006b). Some of the features of the corresponding phase diagram are captured. The Haldane transitions exhibited in the phase diagrams are marked by anomalies in the von Neumann entropy; its maximum at the isotropic point is not related to any critical phenomenon (the system is gapped around such a point), but it is due to the equi-probability of the three spin-1 states occurring at that point (Campos Venuti *et al* 2006b). Since the Berezinskii–Kosterlitz–Thouless transition separating the XY from the Haldane or large- D phases connects a gapless with a gapped regime, it was speculated that an anomaly in the entanglement should highlight such transition (Gu *et al* 2006).

Multipartite entanglement. Although pairwise entanglement allows us to capture the important properties of the phase diagram, it was evidenced that the spin systems are most generically in multipartite entangled state (Wang 2002b, Stelmachovič and Bužek 2004, Bruß *et al* 2005). Just as an example, we point out that the first excited state above a ferromagnetic ground state is a W-state that is a well-known state in quantum information with a multipartite entanglement.

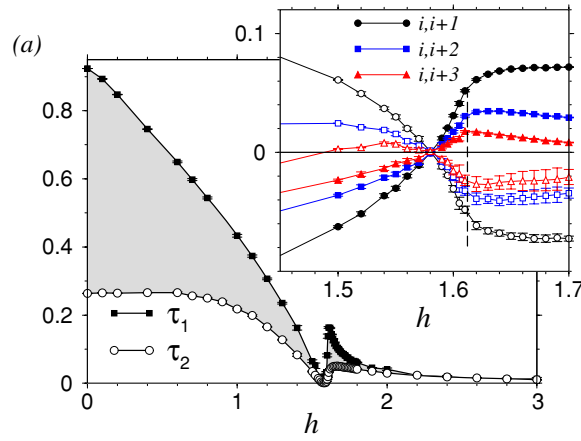


Figure 6. One-tangle τ_1 and the sum of squared concurrences τ_2 as a function of the applied magnetic field in the z -direction for the XYZ model with exchange along y : $J_y = 0.25$ (in unit of exchange along z). Inset: contributions to the concurrence between j th neighbors; full symbols stand for $C^l(j)$, open symbols for $C^{ll}(j)$. The dashed line marks the critical field h_c . (Reprinted with permission from Roscilde *et al* (2004) the American Physical Society, Copyright 2004).

Despite its importance, a *quantitative* description of multipartite entanglement constitutes a challenging problem in the current research. In many-body physics multipartite entanglement has been studied resorting to ‘global’ measures that most often cannot distinguish different types of multipartite entanglement from each other (see however Guhne *et al* (2005)). A first way to estimate multipartite entanglement in a spin system is provided by the entanglement ratio τ_2/τ_1 as the amount of two spins relative to global entanglement. It is interesting to compare the behavior of such quantities for quantum critical and factorizing points of spin models. In fact it emerged that τ_2/τ_1 is small close to quantum critical points. In contrast the entanglement ratio approaches to 1 close to the factorizing point. Close to quantum critical points the entanglement ratio was calculated numerically for $1d$ - XYZ (Roscilde *et al* 2004) (figure 6). The entanglement ratio was calculated close to the factorizing points for the quantum Ising model.

To address multipartite entanglement directly several routes have been suggested. Of interest is the analysis based on geometric entanglement. Most of the works till now concentrated on critical systems. It was proved that geometric entanglement obeys an area law that, most probably, coincides with the well-established area law of the von Neumann entropy (Botero *et al* 2007, Orus 2008a, 2008b, Orus *et al* 2008, Shi *et al* 2009). Such a result provides a further evidence that the universal behavior of the block entanglement close to a critical point traces back to multipartite entanglement.

Although some of the proposed measures rely on n -point correlation functions, there is no clear evidence on whether this is a general feature of multipartite entanglement. An important example of measurement of multipartite entanglement relying on two-point function was proposed in de Oliveira *et al* (2006a, 2006b) and Somma *et al* (2004) as

$$E_{gl}^{(2)} = \frac{4}{3} \frac{1}{N-1} \sum_{l=1}^{N-1} \left[1 - \frac{1}{N-1} \sum_{j=1}^N \text{Tr} \rho_{j,j+l}^2 \right] \quad (17)$$

where $\rho_{j,j+l}$ is the reduced density matrix associated with the sites j and $j+l$. Similarly one can consider also three-body reduced density matrices and construct the corresponding global entanglement measure. Although the precise form has not been established yet, the global entanglement $E_{gl}^{(n)}$, generalization of equation (17) should be related to the set of reduced n -qubit density operators. According to Oliveira and co-workers, then the hierarchy of $E^{(n)}$ might provide a comprehensive description of entanglement in many-body systems already for moderate values of n .

Global entanglement is very sensitive to the existence of QPTs. As a paradigmatic example the authors analyzed the phase diagram in the anisotropy-magnetic field plane (see also Wei *et al* (2005)). By extending an earlier approach developed in Wu *et al* (2004), de Oliveira *et al* also showed how the non-analytic behavior of $E_{gl}^{(n)}$ is related to that of the ground-state energy. Note that from equation (17) it is possible to define an entanglement length proportional to the correlation length ξ . This differs considerably from that one defined by the localizable entanglement (see equation (16)); the latter is always bounded from below by the correlation length and can even be divergent where ξ is finite.

As discussed in Facchi *et al* (2006a, 2006b) and Costantini *et al* (2006), the analysis of the average purity might not be sufficient and the analysis of the distribution of the purity for different partitions could give additional information. Rather than measuring multipartite entanglement in terms of a single number, one characterizes it by using a whole function. One studies the distribution function of the purity (or other measures of entanglement) over all bipartitions of the system. If the distribution is sufficiently regular, its average and variance will constitute characteristic features of the global entanglement: the average will determine the ‘amount’ of global entanglement in the system, while the variance will measure how such entanglement is distributed. A smaller variance will correspond to a larger insensitivity to the choice of the bipartition and, therefore, will be characteristic for different types of multipartite entanglement.

In Patanè *et al* (2007) multipartite entanglement is studied with the aim to shed light on how entanglement is shared in a many-body system. For a quantum XY model the simplest multipartite entanglement of a subsystem made of three arbitrary spins of the chain is considered, then bipartite entanglement between a spin and the other two with respect to all possible bipartitions. It is found that the block of two spins may be entangled with the external spin, despite the latter is not entangled directly with any of the two spins separately (see figure 8). Hence the range of such spin/block entanglement may extend further than the spin–spin entanglement range. It is plausible that increasing the size of the subsystem considered will increase the range of the multipartite entanglement. For instance, the range of spin/block entanglement will increase if we consider a larger block. Hence, a single spin can be entangled with more distant partners, if one allows to cluster them into a large enough block. It would be intriguing to study how spin/block entanglement and, in general, block/block entanglement between subsystems, scale by increasing the size of blocks (with a special look at the phenomenon occurring close to the quantum criticality). We remark that such analysis would be different with respect to the well-known block entropy setting, since in that case one is interested in the block/rest-of-the-system entanglement.

Bound entanglement. We would like to conclude this brief description of the relation between entanglement and magnetic order by analyzing in which cases interacting spin system are in a bound entangled state (see section 3). We shall see that such kind of peculiar entangled states are generically ‘engineered’ by a many-body system at equilibrium as it occurs naturally in certain region of the phase diagram for the ‘last’ entangled states before the complete separability is reached. In this sense the bound entanglement bridges between quantum and

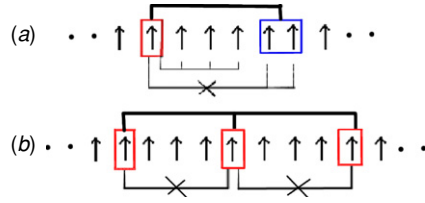


Figure 7. Configurations of spins described in the text and whose entanglement properties are presented in figure 8. We fix $R = 3$; hence, a spin is directly entangled with its first three nearest neighbors. (a) ‘Clustering’ two spins increases the range of entanglement (the scheme is symmetric also for spins on the left of the marked one). (b) Symmetric configuration of spins such that no two-particle entanglement is present, but still the spins share multiparticle entanglement.

classical correlations. Being the bound entanglement a form of demoted entanglement, it appears when quantum correlations get weaker. Bound entangled states were found in both the ground and thermal states of anisotropic XY models (Patanè *et al* 2007, Ferraro *et al* 2008, Cavalcanti *et al* 2008). We follow the approach pursued in Patanè *et al* (2007) where three-spin entanglement in an infinite anisotropic XY chain was analyzed. Two different configurations were considered (see figure 7).

At zero temperature bound entanglement appears (see figure 8) when the spins are sufficiently distant from each other and as in the case of the spin/spin entanglement, it can be arbitrary long ranged near the factorizing field. To prove it, the idea is to resort the ‘incomplete separability’ condition described in the first section. In fact from figure 8 we see that \mathcal{N}_{Ext} may be zero even if $\mathcal{N}_{\text{Centr}}$ is non-zero. Thus, in such a case the density matrix of the spins is PPT for the two symmetric bipartitions of one external spin versus the other two ($\uparrow|\uparrow\uparrow$ and $\uparrow\uparrow|\uparrow$) and negative partial transpose (NPT) for the partition of the central spin versus the other two (we remark that PPT does not ensure the separability of the two partitions for dimensions of local Hilbert space greater than 2). In fact if we were able to distill a maximally entangled state between two spins, then one of two previous PPT partitions would be NPT and this cannot occur since PPT is invariant under LOCC (Horodecki *et al* 1998, Vidal and Werner 2002). Quantum states must be ‘mixed enough’ to be bound entangled. In the ground state a source mixing is the trace over the other spins of the chain. However, if the spins are near enough, the reduced entanglement is free. It results that the effect of the thermal mixing can drive the $T = 0$ free entanglement to bound entanglement (see figure 9). This behavior shown for the Ising model is also found for the entire class of quantum XY Hamiltonians with generic values of anisotropy (the temperature at which the different types of entanglement are decreasing with γ). NPT thermal bound entanglement was also found by Ferraro *et al* (2008), Cavalcanti *et al* (2008) resorting to block entropies. The idea of Acin and co-workers was to calculate the block entropy for two different bipartitions: one in which the two subsystems are made of contiguous block of spins (called the ‘half–half’ partition); the other groups all the spins labeled, say, by even indices in one subsystem and the remaining ones in the other (called the ‘even–odd’ partition). Because of the area law, the entanglement in the even–odd partition is more robust to thermal fluctuations than entanglement in the half–half partition (an hypothesis corroborated by actual calculations by the same authors). Therefore, there is a range of temperatures for which the PPT condition is reached with an even–odd entanglement. The entanglement is bound because single particles cannot distill entanglement (as the half–half bipartition can be singled out to have particles in two different blocks). The calculations are done for a finite set of spins interacting according

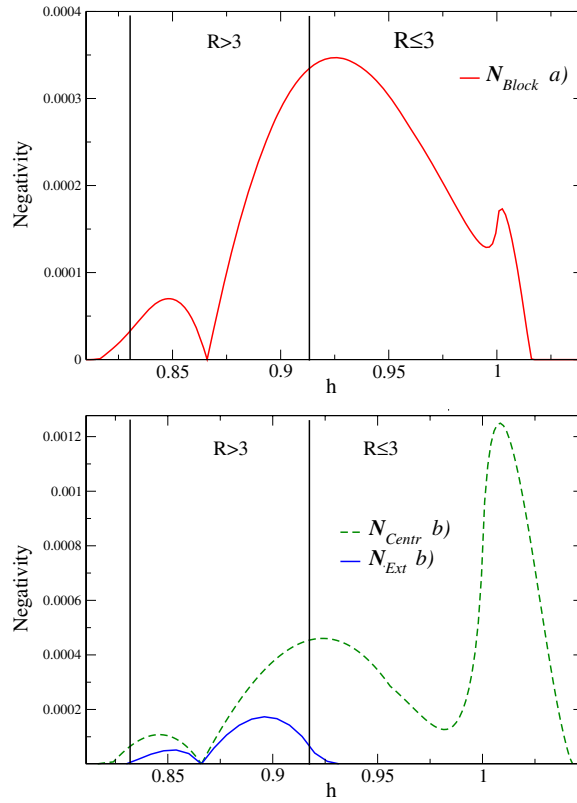


Figure 8. $T = 0$ negativities between one spin and the other two versus magnetic field h are shown for both configurations shown in figure 7. We consider $\gamma = 0.5$. In this case outside the interval marked by the two solid vertical lines the range of spin–spin entanglement is $R \leq 3$ (for values of h inside this interval R grows due to its divergence at factorizing field $h_f = \sqrt{1 - \gamma^2} \simeq 0.86$ (Amico *et al* 2006a)). For configuration (a) (upper panel), \mathcal{N}_{Block} signals the spin/block entanglement for a distance $d = 4$. For values of h outside the vertical lines, the negativity signals genuine spin/block entanglement. For configuration (b) (lower panel), both \mathcal{N}_{Ext} (solid line) and \mathcal{N}_{Centr} (dashed line) are plotted. For values of h outside the vertical lines the three spins share no spin/spin entanglement; hence, for non-zero \mathcal{N}_{Ext} and \mathcal{N}_{Centr} free multiparticle entanglement is present. The latter turns in to bound entanglement for values of h such that only $\mathcal{N}_{Centr} \neq 0$ and $\mathcal{N}_{Ext} = 0$ (both on the left and on the right of the solid lines). Reprinted with permission from Patanè *et al* (2007). Copyright (2007) by IOP Publishing Ltd.

to an isotropic XY Hamiltonian. We remark that such NPT bound entanglement was found in closed systems. Recently it was demonstrated that NPT bound entanglement can arise dynamically from decoherence of multipartite entangled states of GHZ type (Aolita *et al* 2008). Based on that it is intriguing to conjecture that such kind of entanglement could be generated dynamically via decoherence in open systems. Besides NPT, also PPT bound entangled states were found in spin systems at finite temperature (Toth *et al* 2007, 2009). The method developed by Toth *et al* relies on certain relation between entanglement and squeezing of collective spins that can serve as separability test for separability of the given state (like an entanglement witness). The entanglement detected can be of multipartite type despite the relations involve only two-point correlation functions. They considered spin models of the

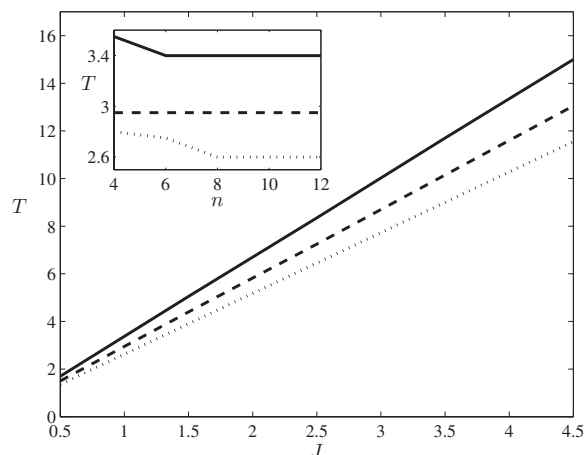


Figure 9. Threshold temperatures above the negativity is zero in the even–odd (solid line), $1 : n - 1$ (dotted line) and half–half (dashed line) partitions. We plot the threshold temperature as a function of the coupling parameter J of the model (1) with $J_x^{(ij)} = J_y^{(ij)} = \delta_{i+1,j}$ and $J^{(ij)} = 0 \forall ij$ for $n = 10$ spins and $h_z = 1.9$. Inset: temperature above which the negativity in the even–odd (solid line), $1 : n - 1$ (dotted line) and half–half (dashed line) partitions is zero as a function of the number n of the spin Hamiltonian (1) with $J = 1$ and $h_z = 1.9$. Reprinted with permission from Cavalcanti *et al* (2008). Copyright by the American Physical Society, © 2008.

Heisenberg type at finite size and proved that a range of temperature exists where their thermal state display multipartite entanglement cannot be distilled for any bipartition of the system.

5. Conclusions and outlook

The use of concepts developed in quantum information science has provided a new twist to the study of many-body systems. Here we presented a specific example of this kind of approach by discussing the relation between magnetism and entanglement. Looking at the next future it seems to us that the most challenging problems are a wider characterization of the multipartite entanglement and, in our opinion most important, a connection between this acquired knowledge and new experiments. Remarkable impact of quantum information in condensed matter has been proving on the possibility of designing more efficient classical numerical algorithms for quantum many-body systems.

As for experimental tests on entanglement in many-body systems, we observe that the most direct method seems relying on the entanglement witnesses that have been derived using thermodynamical quantities. Nevertheless, methods based on neutron scattering techniques on magnetic compounds, that are of particular relevance for spin systems, are also valuable especially for a direct quantification of entanglement in macroscopic systems. In this context we note that more refined experimental analysis seems to be required to extract entanglement. These might disclose new unexplored features of entangled many-body states.

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